Temperature scaling, glassiness and stationarity in the Bak-Sneppen model

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Abstract. We show that the emergence of criticality in the locally-defined Bak-Sneppen model corresponds to separation over a hierarchy of timescales. Near to the critical point the model obeys scaling relations, with exponents which we derive numerically for a one-dimensional system. We further describe how the model can be related to the glass model of Bouchaud (J. Phys. I France **2**, 1705 (1992)), and we use this insight to comment on the usual assumption of stationarity in the Bak-Sneppen model. Finally, we propose a general definition of self-organised criticality which is in partial agreement with other recent definitions.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 05.65.+b Self-organized systems – 45.70.Ht Avalanches

1 Introduction

The concept of *self-organised criticality* or SOC was originally devised by Bak, Tang and Wiesenfeld as an explanation for the apparent ubiquity of scale invariant systems in nature [1–6]. Working on the assumption that such systems are 'critical' in the sense of a continuous phase transition, Bak *et al.* proposed the existence of a class of non-equilibrium models that become critical purely under their own dynamics. This contrasts with other equilibrium and non-equilibrium systems, where at least one control parameter must be set to a particular value before the critical state is attained [7,8], which seems unlikely in the absence of human intervention. Despite these early claims, it soon became clear that many SOC models *do* in fact require parameter tuning, but they had been defined in such a way that the tuning had been carried out *implicitly*. For instance, the sandpile model, which is the canonical SOC system, is usually described in a way that implicitly assumes an infinitesimal driving rate; for finite driving, the model is no longer critical [9,10]. The admission of implicit parameter tuning has allowed the sandpile model to be related to more conventional non-equilibrium systems, and for analytical techniques previously established in other fields to be employed (see *e.g.* [11–13])

Recently, attention has focused on implicit parameter tuning in a subclass of SOC systems known as *extremal dynamical systems*, so-called because they are driven at the location of the minimum (or maximum) of some spatially varying quantity [14,15]. It has already been observed independently by Sneppen [16] and

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Vergeles [17] that the Bak-Sneppen model, which is the simplest and perhaps best understood extremal dynamical system [14,18,19], implicitly assumes that a single temperature-like parameter is set arbitrarily close to zero. However, in our view the mechanism underlying this process was never clearly identified, nor was the inevitability of the parameter tuning properly stressed.

The aims of this paper are fourfold. Firstly, we attempt to provide as clear an explanation as possible that implicit parameter tuning in the Bak-Sneppen model, rather than just being possible, is in fact *inevitable* if one is to have a well-defined physical system. By 'well-defined' we mean that each element evolves according to the state of only a finite number of other elements, as opposed to the global driving rule in the model's original invocation. This relates to recent work on the sandpile model [9], but was never properly discussed in the previous work on this model [16,17]. Note that throughout this paper we follow [9] in refering to the setting of a parameter to zero as 'fine tuning,' although it could be argued that this is just an absolute separation of temperature scales rather than genuine fine tuning. Secondly, we identify the fundamental mechanism underlying this process to be a hierarchy of timescales that diverge relative to each other, which is similar but qualitatively different to the separation of timescales required in other SOC models [20]. Thirdly, we show that the model can be related to a phenomenological glass model of Bouchaud [21], and argue that the Bak-Sneppen model does not reach a statistical steady state in low dimensions, in contrary to the widely held belief that it reaches stationarity after an 'extensive transient' [18]. Finally, we use the insight gained from our work to propose a new way of categorising self-organised criticality, and comment on its relationship with other recent definitions.

This paper is arranged as follows. In Section 2 we carefully reconstruct the Bak-Sneppen model from first principles, and show how a properly defined model with only local interaction rules demands implicit parameter tuning, as in the sandpile model [9]. In Section 3 we show that at finite temperatures the model can be related to a simple glass model, and argue that even at its critical point, the model does not reach a steady state. A mean field description similar to that adopted by Bouchaud is described and solved in Section 4, and we comment on the true meaning of self-organised criticality Section 5.

2 Reconstruction of the Bak-Sneppen model

We begin by reconstructing the Bak-Sneppen model from first principles, following the original arguments in [18]. Although some aspects of this have already been touched upon [16,17], the emphasis here is to demonstrate *utterly unambiguously* that the implicit parameter tuning is unavoidable, rather than just possible.

The local Bak-Sneppen model is defined as follows. The system consists of N elements, each of which is assigned a *barrier* E_i, $i = 1 \ldots N$. In the model's original biological context the E_i represent *fitness* barriers, but they could just as easily correspond to energy barriers on a potential energy landscape, for example. The values of the barriers are drawn from the time-independent *prior distribution* $\rho(E)$, which is assumed to have no delta-function peaks so that there is a vanishing probability of two different elements having the same value of E. The system evolves according to two rules. Firstly, each element becomes *activated* at a rate

$$\mathrm{e}^{-E_i/T}, \qquad (1)$$

where the constant parameter $T > 0$ has the same units as E. An activated site i is assigned a new barrier E_i drawn from $\rho(E)$, corresponding to a shift to a new metastable state with a new barrier height. Note that the activation rate for any given element is *independent* of the state of the rest of the system, so that this activation rule is strictly *local*.

Secondly, for every activated element, another z are chosen and also assigned new barrier values. This interaction term mimics some form of strong coupling between the elements, in the sense that one element changing state drastically alters the *barriers* of z other elements. The way in which the z interacting elements are chosen depends upon the spatial structure of the system. If the elements are arranged on a regular lattice, then the most common rule is to update the barriers of all of the nearest neighbours of the activated element. Hence z is just the lattice coordination number. Alternatively, the z interacting elements may be chosen at random from the remaining $N-1$, with the connections between the elements being randomised anew for every activation event. In this case the system has no spatial structure and the symbol K has often been used, where $K = z + 1$ [22,23].

The system behaviour changes qualitatively as the single parameter $T > 0$ is varied. These different regimes are discussed in turn below.

$T \to 0^+$: In the limit of infinitesimal T, the activation rates $\mathrm{e}^{-E_i/T}$ for different E_i diverge relative to each other. That is, the element with the smallest barrier will become active on one timescale, the one with the second smallest barrier becomes active on another, much longer timescale, and so on. Thus with probability one the first element to become active will be that with the smallest barrier (which is always unique for a finite set of non-degenerate $\{E_i\}$). This is the way in which the Bak-Sneppen model is usually defined; indeed, this *"exponential separation of timescales"* [18] was originally used to justify the extremal dynamics. Although separated timescales arise in other SOC models [20], to our knowledge this is the first time a *hierarchy* of timescales has been explicitly demonstrated.

T small but finite: The strict separation of timescales is lost for finite T and every element has a non-vanishing probability of being the first to become active, so the dynamics are no longer extremal. We now demonstrate that the model is not critical in this regime. This claim is supported by the results of numerical simulations described below, which in turn are supported by mean field analysis.

Let p_i denote the probability that element i becomes active before any other element in the system. Since the time until activation follows an exponential distribution with mean $\mathrm{e}^{E_i/T}$, it is straightforward to show that

$$p_i = \frac{\mathrm{e}^{-E_i/T}}{\sum_{j=1}^{N} \mathrm{e}^{-E_j/T}}, \qquad (2)$$

which obeys $\sum_{i=1}^{N} p_i = 1$. In this notation, the $T \to 0^+$ limit corresponds to $p_{i^*} \to 1$ for the element i^* with the smallest barrier, and $p_j \to 0$ for all $j \neq i^*$. However, $0 < p_i < 1$ for all i when T is finite.

The algorithm employed in the simulations was as follows. N elements were placed on a one-dimensional lattice with periodic boundary conditions. Each element was initially assigned a barrier drawn from the uniform prior distribution $\rho(E) = \{1 \text{ for } 0 \leq E \leq 1, 0 \text{ otherwise}\}$, although we expect the same qualitative behaviour for other $\rho(E)$. For every iteration step, a single element was made active according to the probabilities p_i given in (2). The active element and both of its nearest neighbours were then assigned new barriers, so the number of interacting elements $z = 2$ here. The timescale t was normalised to N activations per unit t, which differs from the usual Bak-Sneppen timescale only by the constant factor N. Distributions were not measured until the mean barrier height $\langle E \rangle = \frac{1}{N} \sum_{i=1}^{N} E_i$ appeared to reach a steady value when plotted against $\log_{10} t$. Note that this is not a rigorous criterion for convergence and we cannot rule out the possibility that long-range spatial correlations may still be growing.

To decide whether or not the system is critical for any given value of T, we employed the usual method of extracting the spatial and temporal correlations from the simulations and checking to see if their tails are consistent with
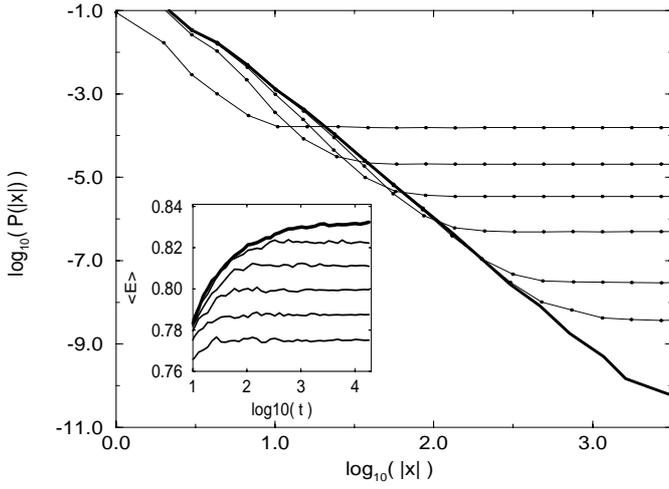
Fig. 1. Log-log plot of the distribution of the jump size $|x|$ between successive active sites for an $N = 10^4$ ring. From bottom to top on the right hand side, the lines refer to $T \to 0^+$ (thick line), 0.001, 0.002, 0.005, 0.01, 0.02 and 0.05 (thin lines), respectively. The thick line has a slope of -3.20 ± 0.05. (Inset) The mean barrier $\langle E \rangle$ against $\log_{10} t$ for $T = 0^+$ (thick line), 0.01, 0.02, 0.03, 0.04 and 0.05 (thin lines, top to bottom). Each unit t corresponds to N activations. Data collection commenced at $\log_{10} t = 4$ in this and all subsequent figures.

Fig. 2. Log-log plot of the limiting value $\lim_{|x| \to \infty} P_{\text{jump}}(|x|)$ as a function of T for $N = 10^4$. The dashed line has a slope of 3.

a power law fit. Apart from complications arising due to finite size effects, anything other than power law behaviour signifies a characteristic scale and a non-critical system. A convenient measure of spatial correlations for this model is the number of lattice sites between two successive active elements. The distribution of these 'jump sizes' $|x|$ for different T are plotted in Figure 1. For $T \to 0^+$ we find that $P_{\text{jump}}(|x|) \sim |x|^{-\pi}$ with $\pi = 3.20 \pm 0.05$, in accord with the known value 3.23 ± 0.02 [14]. However, $P_{\text{jump}}(|x|)$ levels out at a constant T-dependent value $P_{\text{jump}}(\infty) \sim A(T)$ for finite T, indicating that the system is not critical. As demonstrated in Figure 2 for an $N = 10^4$ system, $A(T) \sim T^\alpha$ with $\alpha = 3.0 \pm 0.2$, showing that the critical point is indeed at $T \to 0^+$. This is consistent with the results of Sneppen [16], but our data appears to be much smoother, allowing for a more precise evaluation of the exponent.

The temporal correlations are quantified by $P_{\text{ret}}(t)$, the distribution of times t since the currently active site was last active, which is plotted in Figure 3. Again the data for finite T is clearly not power law, and furthermore the data for small T obeys a scaling function of the form $P_{\text{ret}}(t) \sim t^{-\alpha} \psi(T^\beta t)$ with $\psi(y) \to (\text{const})$ as $y \to 0$, with the exponents $\alpha = 1.58 \pm 0.01$ and $\beta = 3 \pm 0.1$, as demonstrated in the inset of Figure 3.

We interpret the loss of criticality for finite T as follows. For $T \to 0^+$ the location of the active site jumps around the system in the highly correlated manner characteristic of the critical state. By contrast, when T is finite there is a non-zero probability that elements *arbitrarily far* from the active site will become activated at the next time step. Thus the active site can make large jumps to
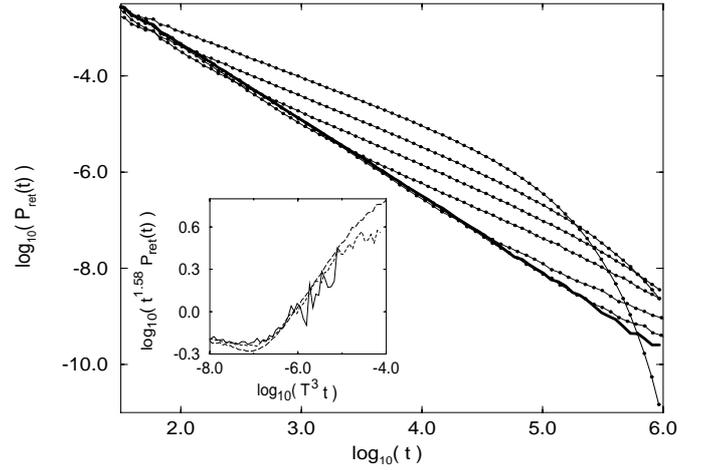
Fig. 3. Log-log plot of $P_{\text{ret}}(t)$, the distribution of times since the currently active site was last active, for different T on an $N = 10^4$ ring. From bottom to top in the middle of the graph, the lines refer to $T \to 0^+$ (thick line), 0.001, 0.002, 0.005, 0.01, 0.02 and 0.05 (thin lines), respectively. The thick line has a slope of -1.58 ± 0.02. (Inset) Scaling plot of $t^{1.58} P_{\text{ret}}(t)$ against $T^3 t$ for $T = 0.001$ (solid line), 0.002 (dotted line) and 0.005 (dashed line).

uncorrelated regions of the system, which we relate to the loss of criticality.

$T = \mathcal{O}(\bar{E})$: Although there is no longer any question of criticality far from $T = 0^+$, it is possible to relate the model to a glass model for finite T. This will be discussed fully in Section 3.

$T \rightarrow \infty$: In this limit, every element has the same activation probability $p_i = 1/N$ independent of E_i, so $P(E, t) \equiv \rho(E)$ for all t.

3 Mean field analysis

In this section we show how the results from the one-dimensional simulations are in qualitative agreement with the solution to the mean field model. The mean field model described here is essentially that of Bouchaud *et al.* with an extra interaction term [25].

Let $P(E, t)\mathrm{d}E$ be the proportion of barriers in the range $[E, E + \mathrm{d}E)$. By adopting the usual mean field simplification of random nearest neighbours, it is then straightforward to show that $P(E, t)$ evolves in time according to [26]

$$\frac{\partial P(E, t)}{\partial t} = -\frac{\mathrm{e}^{-E/T}}{\int_0^\infty \mathrm{e}^{-E'/T} P(E', t)\,\mathrm{d}E'} P(E, t)$$
$$- z\,P(E, t) + (z+1)\rho(E)\,. \quad (3)$$

This equation can be justified by noting that $P(E, t)$ decreases when an element changes its barrier value, which occurs either when it becomes active, or when it is selected as one of the z interacting elements. These two processes are described by the first and second terms on the right hand side of (3), respectively, where the prefactor to the first term is just the continuum analogue of (2). Conservation of probability is ensured by the third term, which corresponds to the $z + 1$ new barriers drawn from $\rho(E)$. We note that the treatment of Veregles is similar [17] but with a poorly defined timescale, resulting in factors of N remaining even *after* taking the continuum limit.

We have solved (3) for the uniform $\rho(E)$ in the limit $t \rightarrow \infty$. The expression for general T is not very instructive, but for small T it simplifies to

$$P(E, \infty) \approx \frac{z+1}{z}\left(1 + \mathrm{e}^{-(E-E_\mathrm{c})/T}\right)^{-1}, \quad (4)$$

with $E_\mathrm{c} = \frac{1}{z+1}$. As $T \rightarrow 0^+$ the exponential in (4) either blows up or decays depending on whether E is less than or greater than E_c, respectively. Thus $P(E, \infty)$ converges to the step function $\frac{z+1}{z}\theta(E - E_\mathrm{c})$, in accord with the known solution of the mean field Bak-Sneppen model [17, 22–24]. However, there is no such discontinuity for finite T and $P(E, \infty)$ is smoothly varying for all $0 < E < 1$, in qualitative agreement with the simulation results in Figure 4 (although note that E_c is larger in the one-dimensional case). Note that (3) can also exhibit glass-like behaviour for $T < 1$, as fully described in [26].

4 Glassiness and the assumption of stationarity

Although the model studied here was described as the Bak-Sneppen model extended to finite temperatures, it
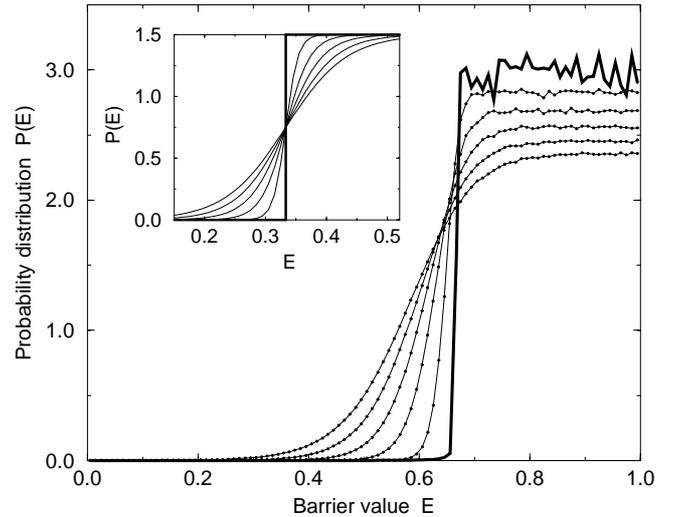
Fig. 4. The barrier distribution $P(E)$ for the same runs as in Fig. 1. From top to bottom on the right hand side, the lines refer to $T = 0^+$ (thick line), 0.01, 0.02, 0.03, 0.04 and 0.05 (thin lines), respectively. (Inset) The corresponding mean field predictions from (4), with $z = 2$.

can equally be viewed as the simple *glass* model of Bouchaud with an extra interaction term [21, 25–27]. The mapping to Bouchaud's model is achieved by the two stage process of first 'switching off' the interactions, *i.e.* setting $z = 0$, and then mapping to a timescale τ which obeys

$$\frac{\partial t}{\partial \tau} = \int_0^\infty \mathrm{e}^{-E'/T} P(E', t)\,\mathrm{d}E'. \quad (5)$$

The mean field equation (3) then becomes

$$\frac{\partial P(E, \tau)}{\partial \tau} = -\mathrm{e}^{-E/T} P(E, \tau)$$
$$+ \rho(E)\int_0^\infty \mathrm{e}^{-E'/T} P(E', \tau)\,\mathrm{d}E', \quad (6)$$

which is the master equation to Bouchaud's model [27]. This alternative interpretation becomes particularly relevant for values of T comparable to the expected barrier height $\bar{E} \equiv \int E\rho(E)\,\mathrm{d}E$, as it is in this regime that Bouchaud's model predicts a glass transition for a certain class of $\rho(E)$.

Since we have already described the relationship between these two models in some detail elsewhere [26], we will not repeat those results here. However, it highlights what may be a very serious problem common to almost all previous treatments of the Bak-Sneppen model. In short, we believe there is already sufficient evidence that the Bak-Sneppen model does *not* reach a statistical steady state, just as Bouchaud's model does not reach stationarity for low temperatures. Clearly this is contrary the the widely held assumption that it does, so this point merits further discussion.

It is widely known from glass theory that, to truly test a system's stationarity, it is not sufficient to measure functions of only one time variable. Instead one must measure

a *two-time correlation function* $C(t+t_{\mathrm{w}}, t_{\mathrm{w}})$, which is some measure of correlation between the state of the system at times t_{w} and $t_{\mathrm{w}} + t$, and show that it does *not* depend on the waiting time t_{w} for large t_{w} [28,29]. Within the current context, almost all of the data commonly recovered from simulations of the Bak-Sneppen model (critical exponents, $P(E, t)$, etc.) are all functions of at most one time variable. For example, simply observing that $P(E, t)$ appears to approach a limiting distribution $P_{\infty}(E)$ does *not* show that the system has become stationary. Nonetheless it appears that criteria similar to this are usually employed to check for a statistical steady state.

To our knowledge, there has only been one instance when two-time correlations have been actively searched for in the Bak-Sneppen model, and this is the numerical work of Boettcher and Paczuski in one and two dimensions [30]. Remarkably, they found that the system behaviour clearly depends on t_{w} for *all* t_{w} they measured. This situation is commonly referred to as *aging* and is a clear indicator of non-stationarity (*ie.* loss of time translational invariance). That aging implies non-stationarity is trivial; the only question that remains is, are Boettcher *et al.*'s results asymptotic, or is stationarity recovered at some very late time τ which is beyond attainable simulation times? Clearly this can never be answered by numerical simulations alone and some form of analytical treatment would be desirable. However, we have shown elsewhere that the mean field model does *not* exhibit aging [26], so analysis would have to limited to the difficult case of finite dimensional systems. This is an important issue whose resolution may help guide attempts to find an exact solution to the model, and further work would be desirable.

5 Discussion and summary

It has been suggested in [9] that SOC systems correspond to absorbing state phase transitions reached in the limit of infinitesimal driving. Although this is almost certainly true, in our view this only explains why SOC systems are critical, *not* why they are self-organised. That is, it does not address, in sufficiently general terms, how a model can be placed at a particular point on its phase diagram without explicit parameter tuning. Note that this is a separate issue as to whether or not the point also happens to be a critical point. An alternative definition of SOC has addressed the self-organising process on more general terms [31,32], but gives little insight into what class of systems need to be critical to reach a statistical steady state.

In light of the work presented in this paper, we now propose a broad definition of SOC that is relevant to the phase diagram approach adopted in [9] and throughout this paper. We suggest that SOC corresponds to that class of models that have a critical point at a *privileged* point on their phase diagrams. By 'privileged' we mean any point at which every parameter takes a value that has some special physical significance. For instance, a positive definite

parameter such as the rate of driving or temperature has two privileged values, 0^{+} and ∞, and indeed all implicit tuning identified thus far do seem to require infinitesimal driving or temperature [15,20]. Similarly a conservation parameter has special points corresponding to 100% conservation and 100% dissipation. Note that such points are *scale invariant* in that they do not depend upon the chosen scale, *i.e.* they are absolute rather than relative points. If any one of these points also happens to be a critical, then it is conceivable that the model could be placed at its critical point 'by accident' and thus be erroneously referred to as 'self-organised' critical. It is becoming increasingly clear that this is precisely the case is all models currently referred to as SOC.

In summary, we have elucidated the mechanism behind the approach to the critical point in the local Bak-Sneppen model. The underlying feature is the existence of a hierarchy of timescales that become separated as a single control parameter approaches zero. We have used the insight gained to propose a new definition of SOC that encompasses all cases of implicit parameter tuning observed so far. Furthermore we have suggested that the Bak-Sneppen model may be non-stationary. This is based on the relationship with a glass model for finite T, and suggests that other SOC models may also exhibit interesting behaviour far from their critical points. We welcome study of these and related questions.

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